TA: LEE, Yat Long Luca
Email: yllee@math.cuhk.edu.hk
Office: Room 505, AB1
Office Hour: Send me an email first, then we will arrange a meeting (if you need it).

1 Midterm 1 Review

The tutorial notes I made basically summarized what you have learnt in the lectures since week 1, but I will given an even quicker summary of topics you have learnt here as a review for midterm 1:

1. Definition of Abstract Vector Spaces
   - Given $(V, +, \times)$, you need to know how to show that it is a vector space, or show that it is not.
   - Given a subset W of a vector space V, you need to know how to show that it is a subspace, or show that it is not.
     - Review some tricks from Tutorial 1. Say, to show that W is a subspace, it suffices to show that for all $a \in F$ and $x, y \in W$, $0 \in W$ and $ax + y \in W$.
   - Understand what does it mean by a "vector". An element is called a "vector" if its underlying set is a vector space. Say, a matrix can be a "vector"; a polynomial can be a "vector"; a differentiable function can be a "vector".
   - Quotient space
   - Direct sum (review tutorial notes and homework 1).
2. Span
   - Given a subset S, $\text{span}(S)$ is the set of all linear combinations of vectors in S. Review lecture videos for the properties of $\text{span}(S)$.
   - Understand what does it mean by $x \in \text{span}(S)$.
3. Linear Independence
   - Given a set of vectors, you need to
     - know how to prove the given set is linearly independent;
     - understand what does it mean by a set is linearly independent.
   - To prove a set is linearly independent, try assuming the set is linearly dependent and come up with a contradiction.
4. Basis
   - Replacement Theorem.
   - Dimension of a vector space
5. Linear Transformation
   - Given $T : V \longrightarrow W$, you need to know what does it mean by T being linear.

- Kernel and Image of a linear map (i.e. Null space and Range)
- Dimension of kernel and image.
  - Dimension Theorem: Let $T : V \longrightarrow W$ be a linear map and V be finite dimensional. Then $\dim V = \dim \ker T + \dim \operatorname{im} T$.
- Injective, Surjective, Bijective

6. Matrix Representation of linear maps
   - Ordered Basis
   - Understand how a vector $x \in V$ can be regarded as a column vector $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in F^n$.
     - That is, understand the map $[\cdot]_\beta : V \longrightarrow F^n$.
7. Composition of linear maps
   - How to read commutative diagram (I do not suggest proving with commutative diagram unless you can justify every arrows are well-defined and the diagram is commutative, but commutative diagram helps locating the object you want to prove).
8. Invertibility
   - Definition of invertibility.
   - $T : V \longrightarrow W$ is invertible if and only if $\dim V = \dim \operatorname{im} T$.
9. Isomorphism [This note]
   - Definition of isomorphism
   - Meaning of "V is isomorphic to W".

Try doing the practice problems suggested by Prof Wu to better familiarize yourself with the material. If you need office hour, please feel free to contact me via email.

2 Isomorphism

Definition 2.1. Let V and W be vector spaces. Then V and W are said to be *isomorphic* if there exists an invertible linear map $T : V \longrightarrow W$ between them.

Q1: Isomorphism is an equivalence relation

Recall that a binary relation $\sim$ is an equivalence relation on a set X if the following holds:

- (Reflexive) $x \sim x$ for all $x \in X$;
- (Symmetric) $x \sim y$ if and only if $y \sim x$ for all $x, y \in X$;
- (Transitive) $x \sim y$, $y \sim z$ implies $x \sim z$ for all $x, y, z \in X$.

Now, we denote "V is isomorphic to W" by $V \cong W$. Show that $\cong$ is an equivalence relation on the class of vector spaces.

Useful Facts

- Let V and W be finite dimensional vector spaces over the same field F. Then $\dim V = \dim W$ if and only if V is isomorphic to W.
- (As a corollary) V is isomorphic to F^n if and only if $\dim V = n$.
- Let V and W as above with dimension n and m, respectively and let β and γ be ordered bases for V and W, respectively. Then the function $\Phi_\beta^\gamma : \mathcal{L}(V, W) \longrightarrow M_{m\times n}(F)$, defined by $\Phi_\beta^\gamma(T) = [T]_\beta^\gamma$ for $T \in \mathcal{L}(V, W)$, is an isomorphism.

3 Suggestions on Exercises

Again, try doing the practice problems suggested by Prof Wu in each HW, I mean those that are not required to hand it.

If you have already done those problems, you can take a look at the book **Advanced Linear Algebra by Steven Roman**, you can download the book legally via the website of springer using your CUHK credentials.

Please feel free to ask us any question before the midterm. Office hour will be held upon request.